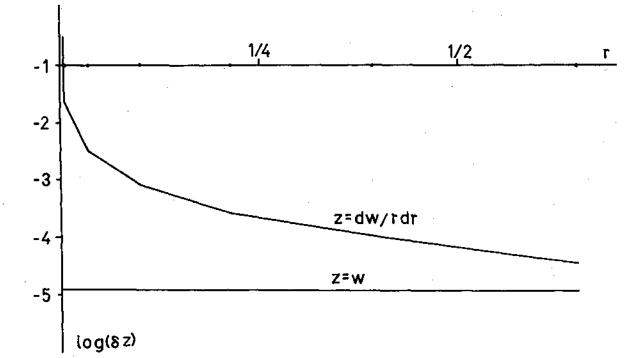

Fig. 2 Clamped circular plate point loaded at the center and discretized with quadratically varying seven elements. Figure shows distribution of errors in w and dw/rdr along the radius r.

pointwise error both in the displacements and strains. Figure 2 depicts the error δw in the displacements w, and the error $\delta(dw/rdr)$ in the circumferential strain, along the radius r of the plate. Note that the error in the strain is confined to the element nearest to the singularity.

In the previous example the exact solution was of the form $r^2(1 + \log r)$. With the cubic ($p = 3$) elements used there is no errors in approximating r^2, the only errors being due to the approximation of $r^2 \log r$. In the more general case, the finite element trial function must approximate both the regular and singular components of the solution. In such a case, the mesh distribution is composed of a uniform (or any better) mesh for approximating the regular portion of the solution plus an exponentially varying mesh for the singular portion.

A problem of this type is provided by a vibrating plate with a point mass at the center.[7] The exact nature of the singularity is not readily available, but since this problem is so closely related to the previous one, also here the singularity was assumed to require a quadratically varying mesh. This mesh distribution was superposed on a uniform mesh and the convergence of the first eigenvalue λ_1 for a clamped plate with a central point mass 0.1 of the total plate mass, is shown in Fig. 3.

Conclusions

Consider a boundary value problem or an eigenvalue problem of the $2m$th order with a solution including a singularity of the form r^α[or $r^\alpha \log(r)$]. Let this problem be defined in n dimensions and be discretized by finite elements inside which the interpolation (shape) functions include a complete polynomial of degree p. The full rate of convergence can be obtained with polynomial elements if they are spaced along r such that the diameter of the i-th element is given by

$$h_i = hi^z, \quad z = [2(p-\alpha)+3-n]/[2(\alpha-m)+n] \qquad (15)$$

It should not escape ones attention that large mesh ratios produce ill-conditioned matrices.[8]

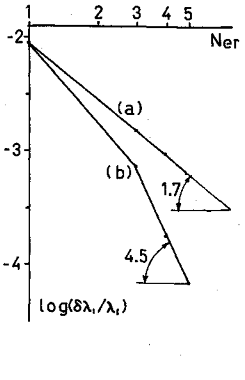

Fig. 3 Clamped circular plate with point mass at the center, ratio of central mass to plate mass being 0.1. Figure shows convergence of the first eigenvalue λ_1 for a) a uniform mesh and b) a quadratically varying mesh superposed on a uniform mesh.

References

[1] Fried, I., "Discretization and Round-Off Errors in the Finite Element Analysis of Boundary Value Problems and Eigenvalue Problems," Ph. D. thesis, June 1971, MIT, Cambridge, Mass.

[2] Fix, G., "Higher Order Rayleigh-Ritz Approximation," *Journal of Mathematics and Mechanics*, Vol. 18, No. 7, 1969, pp. 645–657.

[3] Levy, N., March, P. V., Ostergren, W. J., and Rice, J. R. "Small Scale Yielding Near a Crack in Plane Strain: A Finite Element Analysis," *International Journal of Fracture Mechanics*, Vol. 7, No. 2, June 1971, pp. 143–156.

[3] Levy, N., March, P. V., Ostergen, W. J., and Rice, J. R. "Small Scale Yielding Near a Crack in Plane Strain: A Finite Element Analysis," *International Journal of Fracture Mechanics*, Vol. 7, No. 2, June 1971, pp. 143-156.

[4] Strang, G., "Approximations in the Finite Element Method," 1971, MIT Cambridge, Mass., to be published.

[5] Cowper, R. B., Kosko, E., Lindberg, G., and Olson, M. D., "Static and Dynamic Application of a High Precision Triangular Plate Bending Element," *AIAA Journal*, Vol. 10, No. 10, 1969, pp. 1957–1965.

[6] Fried, I., "Accuracy of Finite Element Eigenproblems," *Journal of Sound and Vibration*, Vol. 18, No. 2, 1971, pp. 289–295.

[7] Roberson, R. E., "Vibration of a Clamped Circular Plate Carrying Concentrated Mass," *Journal of Applied Mechanics*, Vol. 18, No. 4, Dec. 1951, pp. 349–352.

[8] Fried, I., "Condition of Finite Element Matrices Generated from Non-Uniform Meshes," *AIAA Journal*, Vo. 10, No. 2, 1972, pp. 219–221.

Allowable Regions for Stability Multiplier Characteristics

M. K. Sundaresan* and M. A. L. Thathachar†
Indian Institute of Science, Bangalore, India

Introduction

FOLLOWING Popov's[1] innovatory frequency-domain criterion for the absolute stability of systems containing a linear part transfer function $G(s)$ and a nonlinearity in cascade in a negative feedback loop, a number of interesting stability criteria[2–4]‡ have appeared in the control theory literature. All these criteria aim at relaxing the conditions on $G(s)$ corresponding with the imposition of restrictions such as monotonicity, odd property etc. on the nonlinear function and take the following form. For the absolute stability of the system containing a nonlinearity belonging to a certain class $\mathscr{F}_c$, it is sufficient if there exists a function $Z(s)$ belonging to an associated class $\mathscr{Z}_c$ and satisfying the two necessary conditions,

$$Re\, Z(j\omega) \geqq 0 \quad \forall\, \omega \in [0, \infty), \quad Z(j\omega) \not\equiv 0 \qquad (1)$$

$$Re\, Z(j\omega)\, G(j\omega) \geqq 0 \quad \forall\, \omega \in [0, \infty) \qquad (2)$$

A function $Z(s)$ which satisfies Eqs. (1) and (2) is called a"stability multiplier" for the system. However, the applicability of these criteria is largely restricted by the fact that these do neither give a procedure for determining the multiplier $Z(s)$ nor indicate the possibility of the existence of such a function for a given problem. This Note is aimed at removing some of the difficulties involved in the determination of stability multipliers for a given $G(s)$.

Received February 1, 1972; revision received April 12, 1972.
Index categories: Aircraft Handling, Stability and Control; Navigation, Control, and Guidance Theory.

* Research Scholar, Department of Electrical Engineering.
† Assistant Professor, Department of Electrical Engineering.

‡ These are only a few of the prominent results.

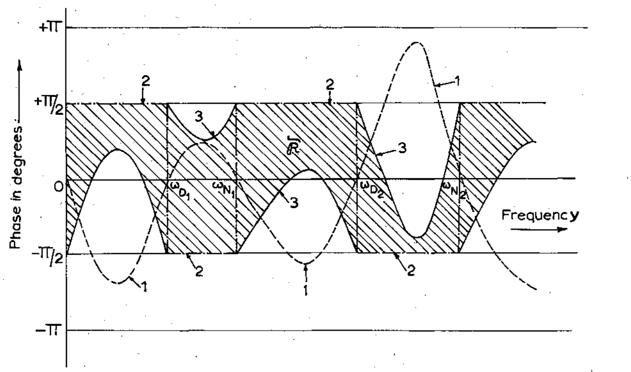

Fig. 1 Illustration of the construction of the allowable region $\bar{\mathscr{R}}$. Curve 1: assumed form of $\Phi(G)$; curve 2: $\Phi(Z_{LC})$; curve 3: $\Phi[(GZ_{LC})^{-1}]$.

Construction of an Allowable Region for Multiplier Phase Characteristics

It is well known that a necessary condition for the absolute stability of a nonlinear system is that, the system with the nonlinearity replaced by a linear element be stable and hence only those systems which are stable on linearization, will be considered. This means that the Nyquist locus of $G(j\omega)$ does not intersect the negative real axis as ω increases from zero to infinity and assures that the phase function arg $G(j\omega)$, hereafter denoted $\Phi(G)$, is uniquely defined for all ω and takes values only in $[-\pi, +\pi]$ as shown in the figure. Brockett and Willems[2] have shown that, in the case of such a system there always exists a multiplier $Z(s) = Z_{LC}(s)$ of the form,

$$Z_{LC}(s) = s^{\pm 1}\pi_i(s^2 + \omega_{N_i}^2)/\pi_i(s^2 + \omega_{D_i}^2) \tag{3}$$

(s^{+1} to be used if the first nonzero value of $\Phi(G)$ is negative and s^{-1} to be used otherwise) and have indicated the procedure for its construction. $Z_{LC}(s)$ has poles and zeros only on the imaginary axis of the s plane and $\Phi(Z_{LC}) = \arg Z_{LC}(j\omega)$ takes the values,

$$\Phi(Z_{LC}) = \begin{cases} +\pi/2 & \text{if} \quad \Phi(G) < 0 \\ -\pi/2 & \text{if} \quad \Phi(G) > 0 \end{cases} \tag{4}$$

and jumps from $\mp\pi/2$ to $\pm\pi/2$ at the frequencies where $\Phi(G)$ crosses the ω axis (see Fig. 1).

Another multiplier which has an important bearing on the further analysis will now be considered. It is easy to see that, if $Z_{LC}(s)$ is a stability multiplier for the system, then $[G(s)\,Z_{LC}(s)]^{-1}$ is also a stability multiplier. Since,

$$\Phi[(GZ_{LC})^{-1}] = \Phi(G^{-1}) + \Phi(Z_{LC}^{-1}) \tag{5}$$

this phase function jumps from $\pm\pi/2$ to $\mp\pi/2$ at the same frequencies where $\Phi(Z_{LC})$ jumps, but in an exactly opposite fashion.

Let

$$\mathscr{R}_{\Phi(Z_1)}^{\Phi(Z_2)}$$

denote the region in the frequency-phase plane, having the phase curve $\Phi(Z_1)$ as the bottom boundary and $\Phi(Z_2)$ as the top boundary and let the ω axis be separated into intervals at the frequencies where $\Phi(G)$ changes sign i.e. $(0, \omega_{D_1})$, $(\omega_{D_1}, \omega_{N_1})$, $(\omega_{N_1}, \omega_{D_2})$ ... as in the figure. Let

$$\mathscr{R}_j = \begin{cases} \mathscr{R}_{\Phi(Z_{LC})}^{\Phi[(GZ_{LC})^{-1}]} & \text{if} \quad \Phi(G) > 0 \\ \mathscr{R}_{\Phi[(GZ_{LC})^{-1}]}^{\Phi(Z_{LC})} & \text{if} \quad \Phi(G) < 0 \end{cases} \tag{6}$$

for the jth interval on the ω axis. It will now be proved that the region

$$\bar{\mathscr{R}} = \cup_j \mathscr{R}_j \tag{7}$$

constitutes the region in which the phase characteristics of all stability multipliers for the given nonlinear system should lie. In other words, it will now be proved that a necessary and sufficient condition for $Z(s)$ to be a stability multiplier is that,

$$\Phi(Z_{LC}) \leqq \Phi(Z) \leqq \Phi[(GZ_{LC})^{-1}] \quad \text{when} \quad \Phi(G) > 0 \tag{8}$$

and

$$\Phi[(GZ_{LC})^{-1}] \leqq \Phi(Z) \leqq \Phi(Z_{LC}) \quad \text{when} \quad \Phi(G) < 0 \tag{9}$$

for all $\omega \in [0, \infty)$. The region $\bar{\mathscr{R}}$ is shown hatched in the figure. *Proof*: if $Z(s)$ is a stability multiplier for the given system, conditions (1) and (2) are equivalent to

$$-\pi/2 \leqq \Phi(Z) + \Phi(G) \leqq +\pi/2 \tag{10}$$

and

$$-\pi/2 \leqq \Phi(Z) \leqq +\pi/2 \tag{11}$$

Also, since the linearized system is assumed to be stable,

$$-\pi \leqq \Phi(G) \leqq +\pi \tag{12}$$

a) *Necessity*: 1) when $\Phi(G)$ is positive,

$$\Phi(Z_{LC}) = -\pi/2 \tag{13}$$

Also, from Eqs. (10) and (11),

$$-\pi/2 \leqq \Phi(Z) \leqq (\pi/2) - \Phi(G)$$

and Eq. (8) follows from Eqs. (5) and (13), 2) when $\Phi(G)$ is negative,

$$\Phi(Z_{LC}) = +\pi/2 \tag{14}$$

Also, from Eqs. (10) and (11),

$$(-\pi/2) - \Phi(G) \leqq \Phi(Z) \leqq +\pi/2$$

and Eq. (9) follows from Eqs. (5) and (14).

b) *Sufficiency*: 1) assume that Eq. (8) holds. Since

$$\Phi(G) > 0, \quad \Phi(Z_{LC}) = -\pi/2$$

and from Eq. (8),

$$\Phi(Z_{LC}) \leqq \Phi(Z) \leqq -\Phi(G) - \Phi(Z_{LC})$$

i.e., $-\pi/2 \leqq \Phi(Z) \leqq -\Phi(G) + \pi/2$; i.e., $-\pi/2 \leqq \Phi(Z) \leqq \Phi(Z) + \Phi(G) \leqq +\pi/2$ since $\Phi(G) > 0$. 2) assume that Eq. (9) holds. Since $\Phi(G) < 0$, $\Phi(Z_{LC}) = +\pi/2$ and from Eq. (9),

$$-\Phi(G) - \Phi(Z_{LC}) \leqq \Phi(Z) \leqq \Phi(Z_{LC})$$

i.e., $-\Phi(G) - \pi/2 \leqq \Phi(Z) \leqq +\pi/2$; i.e., $-\pi/2 \leqq \Phi(G) + \Phi(Z) \leqq \Phi(Z) \leqq +\pi/2$ since $\Phi(G) < 0$. Hence Eqs. (10) and (11) are satisfied for all $\omega \in [0, \infty)$ and $Z(s)$ is a stability multiplier.

Applications of the Construction of $\bar{\mathscr{R}}$

Determination of the nonlinearity class for a stable system

a) It may be realized that with the determination of the allowable region $\bar{\mathscr{R}}$ for a given $G(s)$, the stability problem is reduced to one of constructing a multiplier function $Z(s)$ having the phase characteristic inside $\bar{\mathscr{R}}$. $Z(s)$ can then be analysed to determine the class $\mathscr{Z}_c$ to which it belongs; and hence, the nonlinearity class $\mathscr{F}_c$ for which stability of the feedback system is assured. A few computer-oriented methods for the construction of $Z(s)$ are given in Ref. 5.

b) In a few particular cases, some facts can be drawn by an inspection of the allowable region $\bar{\mathscr{R}}$.

1) If it is possible to draw within $\bar{\mathscr{R}}$, a monotonically increasing (monotonically decreasing) curve with values zero at $\omega = 0$ and $+\pi/2(-\pi/2)$ at $\omega = \infty$, only then, there is a possibility of

the existence of a stability multiplier of the form $(1 \pm qs)$, $q > 0$ and stability can be proved with Popov-type nonlinearities.

2) Let $\mathscr{R}_0^{\pi/2}$ ($\mathscr{R}^0_{-\pi/2}$) denote the strip on the $\omega - \varphi$ plane between the lines $\varphi = 0$ and $\varphi = +\pi/2(-\pi/2)$. Only if either $\tilde{\mathscr{R}} \cap \mathscr{R}_0^{\pi/2}$ or $\tilde{\mathscr{R}} \cap \mathscr{R}^0_{-\pi/2}$ is nonnull for all $\omega \varepsilon [0, \infty)$, then a stability multiplier of the *RL* or *RC* form[2] is possible and stability of the system with monotone nonlinearities can be inferred by constructing this function.

Determination of the allowable sector for monotone nonlinearities

The analysis in the previous section was centered on nonlinearities belonging to the infinite sector. However, for the finite sector problem i.e., where $0 \leqq xf(x) \leqq Kx^2, K < \infty$, the region $\tilde{\mathscr{R}}$ has to be constructed from $\Phi(G + 1/K)$ instead of $\Phi(G)$. It must be noted that the region $\tilde{\mathscr{R}}$ constructed from $\Phi(G + 1/K)$ will be larger than that constructed from $\Phi(G)$ and in many a case, a system for which stability cannot be proved with a nonlinearity of a particular class in the infinite sector, can be proved to be stable with a similar nonlinearity in a finite sector. This enlargement in $\tilde{\mathscr{R}}$ as K is reduced can be used to determine the largest sector within which the nonlinearities of a particular class should lie, for the stability of the system. The procedure for determining the allowable sector $[0, K_M]$ for monotone nonlinearities will now be indicated.

Since the systems considered will be stable with all linear feedback, the Hurwitz sector will be $[0, \infty]$. At the other extreme, the Popov sector $[0, K_P]$ can be determined, as shown by Siljak[6] by plotting the envelope of the Popov inequality

$$Re(1 + jq\omega)\, G(j\omega) + 1/K \geqq 0 \; \forall \; \omega \in [0, \infty)$$

on the $1/K - q$ parameter plane.

Now, K_M should satisfy, $K_P \leqq K_M \leqq \infty$. Hence, by constructing the region $\tilde{\mathscr{R}}$ from $\Phi(G + 1/K)$ for different values of K increased in steps from $K = K_P$, the largest value of K for which either $\tilde{\mathscr{R}} \cap \mathscr{R}_0^{\pi/2}$ or $\tilde{\mathscr{R}} \cap \mathscr{R}^0_{-\pi/2}$ (just described) is nonnull for all $\omega \in [0, \infty)$, is determined. Multipliers which have the phase characteristic within this region of intersection are constructed[5] and are analysed to see if they are of the *RL* or *RC* form. If not, a slightly smaller value of K is chosen and the procedure is repeated. Then, K_M is the largest value of K for which these conditions are satisfied.

The determination of K_M will be very useful in such practical problems as the stability analysis of nuclear reactor control systems, wherein K_M fixes the maximum power level for stable operation[7] of the reactor.

References

[1] Popov, V. M., "Absolute Stability of Nonlinear Systems of automatic Control," *Automation and Remote Control*, Vol. 22, Feb. 1962, pp. 857–875.

[2] Brockett, R. W. and Willems, J. L., "Frequency Domain Stability Criteria - Parts I and II," *IEEE Transactions on Automatic Control*, Vol. AC-10, July 1965, pp. 255–261; also Oct. 1965, pp. 407–413.

[3] Thathachar, M. A. L., Srinath, M. D., and Ramapriyan, H. K., "On a Modified Lur'e Problem," *IEEE Transactions on Automatic Control*, Vol. AC-12, Dec. 1967, pp. 731-739.

[4] O'Shea, R. P., "An Improved Frequency-Time Domain Stability Criterion for Autonomous Continuous Systems," *IEEE Transactions on Automatic Control*, Vol. AC-12, Dec. 1967, pp. 725-731.

[5] Sundareshan, M. K. and Thathachar, M. A. L., "Construction of Stability Multipliers for Nonlinear Autonomous Systems," Rept. EE-16, April 1971, Dept. of Electrical Engineering, Indian Inst. of Science, Bangalore, India.

[6] Siljak, D. D., *Nonlinear Systems—The Parameter Analysis and Design*, 1st ed., Wiley, New York, 1969, pp. 356-360.

[7] Baran, W. and Meyer, K., "Effect of Delayed Neutrons on the Stability of a Nuclear Power Reactor," *Journal Nuclear Science and Engineering*, Vol. 24, 1966, pp. 356–361.

Average Circumferential Pressure on Inclined Bodies of Revolution at Hypersonic Speed

M. D. BENNETT* AND G. E. BARR†

Sandia Laboratories, Albuquerque, N. Mex.

Introduction

A SMALL probe is sometimes fixed to the nose of a missile to permit measurement of the static pressure, and the flight altitude is then inferred from the pressure. Typically, the probe is an axially-symmetric body with several orifices located around the circumference, each connected to a common chamber or manifold where the pressure is sensed. The manifold serves to average the local pressures acting at the orifices so that the measured value is relatively insensitive to angle of attack and roll position. The pressure in the manifold may differ significantly from the average hydrostatic pressure at the probe surface. The extent of the difference depends upon a host of parameters, such as heat transfer and pressure,[1,2] orifice inclination,[3] etc. Further, the pressure along the surface of the probe may depend upon shock-wave boundary-layer interactions which occur when the wave emanating at the juncture of the probe afterbody and the missile nose section coalesces with the probe boundary layer.[4] Finally, the pressure that is to be sensed depends upon the number of orifices and their locations, and that dependence is discussed here for the hypersonic speed regime.

Derivation of Equation

The pressure on the surface of a body in hypersonic flow as specified by the Newtonian impact theory[5] is $C_p = C_{p,o} \sin^2\xi$, where C_p and $C_{p,o}$ are the local and stagnation-point pressure coefficients, respectively, and ξ is the apparent flow incidence angle, the angle between the velocity vector and the plane tangent to the surface at the desired point. If δ represents the local slope of the surface with respect to the longitudinal axis of a body of revolution and α represents the total angle of attack of the body, the angle of incidence becomes $\xi = \arcsin(\cos\alpha \sin\delta + \sin\alpha \cos\delta \sin\Phi)$ where Φ is the angular position of the point along the circumference.

When the points are positioned on the circumference in equal increments so that $n(\Delta\varphi) = 2\pi$ rad, where n denotes the total number of points, the incidence angle at any point $j = 1\ 2, \ldots$ may be expressed $\xi = \arcsin\{\cos\alpha \sin\delta + \sin\alpha \cos\delta \sin[\varphi + (j-1)(2\pi/n)]\}$ where φ denotes the body aerodunamic roll angle. Therefore, the pressure coefficient at location j becomes

$$C_p = C_{p,o}\{\cos\alpha \sin\delta + \sin\alpha \cos\delta \sin[\varphi + (j-1)(2\pi/n)]\}^2 \qquad (1)$$

Since the coefficient of the numerical average of the local pressures is equal to the average of the local pressure coefficients, the average value is simply

$$\langle C_p \rangle = (1/n) \sum_{j=1}^{n} C_p$$

Hence, the equation describing the average pressure is

$$\langle C_p \rangle = (1/n) C_{p,o} \sum_{j=1}^{n} \{\cos\alpha \sin\delta + \sin\alpha \cos\delta \sin[\varphi + (j-1)(2\pi/n)]\}^2 \qquad (2)$$

Received February 22, 1972. This work was supported by the U. S. Atomic Energy Commission.

Index category: Supersonic and Hypersonic Flow.

* Member Technical Staff, Aerodynamics Projects Department. Associate Fellow AIAA.

† Member Technical Staff, Mathematics and Computer Science Department.